velocity of the wave becomes sharper: $d\rho/d\xi|_{\max} \sim (\tilde{u}_0^2 v^2 - 1)^n$. We note that, for a weak wave, taking account of the change in the density of the mobile dislocations gives corrections of the following order of smallness with respect to $(\tilde{u}_0^2 v^2 - 1)$.

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INSTABILITY OF A SPHERICAL BODY UNDER UNIFORM LOADING

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UDC 539.374

Proceeding from the three-dimensional equations of stability theory in the dynamical formulation, the stability of a sphere made out of a reinforced elastic–viscous–plastic material is investigated under uniform loading. The subcritical strains are small. It is shown that the results obtained from approximate and three-dimensional theories for elastic–plastic stability problems differ qualitatively and quantitatively in practice. A similar problem has been discussed earlier in [1] in a static formulation on the basis of an approximate approach and the relationships of the theory of small elastic–plastic strains.

The axisymmetric elastic–plastic state of a spherical body of radii r_1 and r_2 subject to the action of an internal pressure q is determined by the relationships

$$\begin{aligned}
\sigma_r^{p0} &= -q_0 + \frac{4k_0}{2+c_0}\left[6\ln\frac{r}{\alpha} + c_0\gamma^3\left(\frac{1}{\alpha^3} - \frac{1}{r^3}\right)\right],\\
\sigma_\theta^{p0} &= -q_0 + \frac{4k_0}{2+c_0}\left[6\ln\frac{r}{\alpha} + 3 + c_0\gamma^3\left(\frac{1}{\alpha^3} + \frac{1}{2r^3}\right)\right],\\
\sigma_r^{e0} &= 4k_0\gamma^3\left(1 - \frac{1}{r^3}\right) \quad \sigma_\theta^{e0} = 4k_0\gamma^3\left(1 + \frac{1}{2r^3}\right),\\
\alpha &= r_1 r_2^{-1}.
\end{aligned} \tag{1}$$

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Here and in the following all quantities having the dimensions of a stress are divided by the shear modulus μ and of length, by the outer radius r_2; c_0 is the reinforcement coefficient, k_0 is the yield stress, and the subscripts p and e are assigned to the components of the stresses in the plastic and elastic regions, respectively.

The radius of the elastic–plastic boundary γ satisfies the equation

$$\gamma^3\left(1-\frac{c_0}{\alpha^3(2+c_0)}\right)=6\ln\frac{\gamma}{\alpha(2+c_0)}-\frac{q_0}{4k_0}+\frac{2}{2+c_0}. \tag{2}$$

In order to determine the elastic-plastic state of a spherical body, the equilibrium equations, the plasticity condition [2], the general equations of elasticity theory, and also the conjugate conditions of solutions in the elastic and plastic regions, were included.

We write the basic relationships necessary for an investigation of the stability of a spherical body which possesses elastic–viscous–plastic properties. The connection between the amplitude values of stresses, strains, and displacements in the plastic and elastic regions are represented in the form

$$\sigma_{ij}^p=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}-\frac{4\mu^2\left(e_{kl}-\frac{1}{3}e_{mm}\delta_{kl}\right)\left(s_{kl}^0-ce_{kl}^{p0}\right)}{k^2(2\mu+c+s\eta)}\left(s_{ij}-ce_{ij}^{p0}\right),\quad \sigma_{ij}^e=\lambda e_{kk}\delta_{ij}+2\mu e_{ij}, \tag{3}$$

respectively, where

$$e_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}). \tag{4}$$

The conditions on the elastic–plastic boundary γ are of the form

$$n_j\left[\sigma_{ij}+\sigma_{ij,k}X_k\right]=0,\quad \left[u_i+u_{i,k}^0X_k\right]=0, \tag{5}$$

where X_k are the vector components which determine the shape of the elastic–plastic boundary; the square brackets denote the difference between the corresponding components in the plastic and elastic regions.

The equilibrium equations and the boundary conditions [4] for the amplitude values of the displacements are written in the form

$$(\sigma_{ij}+\sigma_{jk}^0u_{i,k})_{,j}+\rho\omega^2u_i=0,\quad (\sigma_{ij}+\sigma_{jk}^0u_{i,k})\,n_j=q_j^0u_{i,j}. \tag{6}$$

It is assumed that the load changes its direction as a result of small perturbations.

The system of equations in spherical coordinates for determination of the displacements in the plastic region can, according to (6), (3), and (4), be written in the form

$$\begin{gathered}
r(4-3a_0)u_{1,1}+\frac{u_{1,33}}{\sin^2\theta}+u_{1,22}+\operatorname{ctg}\theta\, u_{1,2}+2u_1+u_{1,11}\left(1-a_0+\sigma_r^0\right)r^2+\\
+r^2\frac{\partial p}{\partial r}+\sigma_\theta^0\left(u_{1,22}+\frac{1}{\sin^2\theta}u_{1,33}+\operatorname{ctg}\theta\, u_{1,2}+2u_1+4ru_{1,1}\right)+\rho\omega^2u_1=0,\\
r^2\left(1+\frac{a_0}{2}\right)u_{1,12}+r^2u_{2,11}+4ru_{1,2}-2u_2+2u_{2,22}+\frac{1}{\sin^2\theta}(u_{2,33}+u_{3,23}-\\
-2\operatorname{ctg}\theta\, u_{3,3})+\sigma_r^0(u_{2,11}r^2-2ru_{2,1}+2u_2)+r^2\frac{\partial p}{\partial\theta}+\sigma_\theta^0\Big(u_{2,22}+4ru_{1,2}-2u_2-\\
-\frac{1}{\sin^2\theta}u_2+\frac{1}{\sin^2\theta}u_{2,33}-2\operatorname{ctg}\theta\frac{1}{\sin^2\theta}u_{3,3}+\operatorname{ctg}\theta\, u_{2,2}\Big)+\rho\omega^2r^2u_2=0,\\
r^2\left(1+\frac{a_0}{2}\right)u_{1,13}+r^2u_{3,11}+4ru_{1,3}+u_{3,22}+u_{2,32}+\operatorname{ctg}\theta\,(u_{3,2}+u_{2,3})+\\
+\frac{2}{\sin^2\theta}u_{3,33}+r^2\frac{\partial p}{\partial\varphi}+\sigma_r^0(r^2u_{3,11}+2u_3-2ru_{3,1})+\sigma_\theta^0\Big(u_{3,22}+\operatorname{ctg}\theta\, u_{2,3}+\\
+2r(u_{3,1}+u_{1,3})+\frac{1}{\sin^2\theta}u_{3,33}-2u_3\Big)+\rho\omega^2r^2u_3=0,\quad a_0=4(2+c_0+s\eta)^{-1}.
\end{gathered} \tag{7}$$

The incompressibility condition has the form

$$r^2u_{1,1}+u_{2,2}+2ru_1+\operatorname{ctg}\theta\, u_2+\frac{1}{\sin^2\theta}u_{3,3}=0. \tag{8}$$

We determine the displacement in the elastic region from the system (7) and (8), where the value of a_0 should be set equal to zero. In the case of an approximate approach [5, 6] it is necessary to set quantities in (7) equal to zero which take the loading parameter into account, i.e., $\sigma_r^0=\sigma_\theta^0=0$.

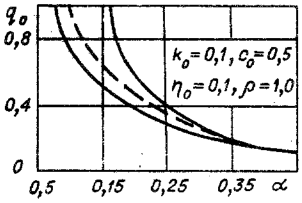

Fig. 1

The solution both in the plastic and in the elastic region is sought in the form of a Fourier series in the spherical functions

$$u_1 = \sum_{n=\nu}^{\infty}\sum_{\nu=1}^{\infty} A_{n\nu}(r)\,Y_{n\nu}(\theta,\varphi),\quad u_2 = \sum_{n=\nu}^{\infty}\sum_{\nu=1}^{\infty} B_{n\nu}(r)\frac{\partial}{\partial\theta}Y_{n\nu}(\theta,\varphi),$$
$$u_3 = \sum_{n=\nu}^{\infty}\sum_{\nu=1}^{\infty} C_{n\nu}(r)\frac{\partial}{\partial\varphi}Y_{n\nu}(\theta,\varphi),\quad p = \sum_{n=\nu}^{\infty}\sum_{\nu=1}^{\infty} D_{n\nu}(r)\,Y_{n\nu}(\theta,\varphi). \tag{9}$$

The functions $Y_{n\nu}$ satisfy the equation

$$\left[\frac{\partial^2}{\partial\theta^2} + \operatorname{ctg}\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} + n(n+1)\right]Y_{n\nu}(\theta,\varphi) = 0.$$

We will omit the subscripts n and ν from here on to simplify the writing.

One can convince oneself that B = C. Actually, differentiating the second equation of (7) with respect to φ and the third with respect to θ and subtracting the equations obtained, we get

$$L(u_{2,3} - u_{3,2}) = 0, \tag{10}$$

where L is some differential operator.

Since (10) is satisfied for any r, it follows from (9) that B = C. Substitution of the expansions (9) into Eqs. (7) and (8) and elimination of B and D gives a fourth-order ordinary differential equation of the form

$$\sum_{m=0}^{4} R_m^p(r)\frac{\partial^m A}{\partial r^m} = 0, \tag{11}$$

where

$$R_0 = (N-2)\left(1 + \sigma_\theta^0 - \rho_0 r^2 N^{-1}\right);$$
$$R_1 = -r\left(4 - 3a_0 + 4\sigma_\theta^0 - rN^{-1}\left(8\sigma_{r,r}^0 + 4\rho_0 r - \sigma_{\theta,r}^0(2+N)\right)\right);$$
$$R_2 = -r^2N^{-1}\Big(2N - \frac{3}{2}a_0N - \rho_0 r^2 + \sigma_r^0 N - \sigma_\theta^0 -$$
$$- 12\left(1+\sigma_r^0\right) - 4r\sigma_{r,r}^0 - 2r\sigma_{\theta,r}^0\Big); \tag{12}$$
$$R_3 = r^3N^{-1}\left(6\sigma_r^0 + r\sigma_{r,r}^0 + 2\sigma_\theta^0 + 8\right);$$
$$R_4 = r^4N^{-1}\left(1+\sigma_r^0\right);\quad N = n^2 + n;\quad \rho_0 = \rho\omega^2.$$

A similar equation occurs in the elastic zone; it is only necessary to set in (12) $a_0 = 0$ and $\sigma_r^0 = \sigma_r^{e0}$, $\sigma_\theta^0 = \sigma_\theta^{e0}$, which have the form of (1).

The boundary conditions (6) are reduced to the form

$$\sum_{i=0}^{3} Q_i\frac{\partial^i A}{\partial r^i} = 0,\quad r^2\frac{\partial^2 A}{\partial r^2} + 2r\frac{\partial A}{\partial r} + A(N-2) = 0 \tag{13}$$

at r = 1, α.

Here

$$Q_0 = 2\rho_0 r,\quad Q_1 = 6 + 8\sigma_r^0 - \sigma_\theta^0(2+N) - 3N + \frac{3}{2}a_0N,$$
$$Q_2 = r\left(6 + 4\sigma_r^0 + 2\sigma_\theta^0\right),\quad Q_3 = r^2\left(1+\sigma_r^0\right). \tag{14}$$

For r = 1 one should set in (14) $a_0 = 0$, $\sigma_r^0 = \sigma_r^{e0}$, and $\sigma_\theta^0 = \sigma_\theta^{e0}$, and at r = α, $\sigma_r^0 = \sigma_r^{p0}$ and $\sigma_\theta^0 = \sigma_\theta^{p0}$.

Conditions (5) in terms of spherical functions are as follows:

$$[A] = 0,\quad \left[\frac{\partial A}{\partial r}\right] = 0,\quad \left[\frac{\partial^2 A}{\partial r^2}\right] = 0,\quad r^2N^{-1}\left(1+\sigma_r^{e0}\right)\left[\frac{\partial^3 A}{\partial r^3}\right] + a_0\frac{\partial A}{\partial r} = 0. \tag{15}$$

In the case of the approximate approach equations similar to (13) can be derived on the basis of the boundary conditions [5, 6]

$$(\sigma_{ij} + \sigma^0_{ik} u_{k,j}) n_j = q^0_j u_{i,j}.$$

We will express the pressure q_0 from Eq. (2) everywhere so that q_0 enters the system (11)-(15) implicitly through the critical radius γ. A nontrivial solution of the system (11)-(15) corresponds to stability loss of the sphere.

Let us replace by the method of finite differences the equations being investigated with a system of homogeneous linear algebraic equations whose determinant depends on the parameters of the medium c_0, k_0, and η_0, the complex number $s = i\omega$, the wave-formation parameter n, the dimensionless radius α, and the critical radius γ.

Equating the determinant of the system to zero, we find the conditions of stability loss of the sphere. A peculiarity of the numerical application consists of the fact that here γ takes a finite number of values, running through the entire interval from α to 1. Consequently, one can find the minimum root γ_{cr} of the determinant, which corresponds to the critical pressure q_0. The calculations were performed on an M222 computer. The dependence of the critical pressure q_0 on the geometry of the construction α is presented in Fig. 1 for $k_0 = 0.1$, $c_0 = 0.5$, $\eta_0 = 0.1$, $\rho = 1$, and $n = 2$ ($n = 0$ and $n = 1$ are excluded from the analysis of the numerical application, since they have no physical meaning).

The lower curve corresponds to the three-dimensional theory alternative, the dashed curve corresponds to the approximate approach, and the upper curve corresponds to the pressure at which the entire sphere is in the plastic state (exhaustion of carrying capacity). Comparison shows that the results obtained from exact and approximate theory differ only quantitatively and insignificantly besides.

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